Lecture 19

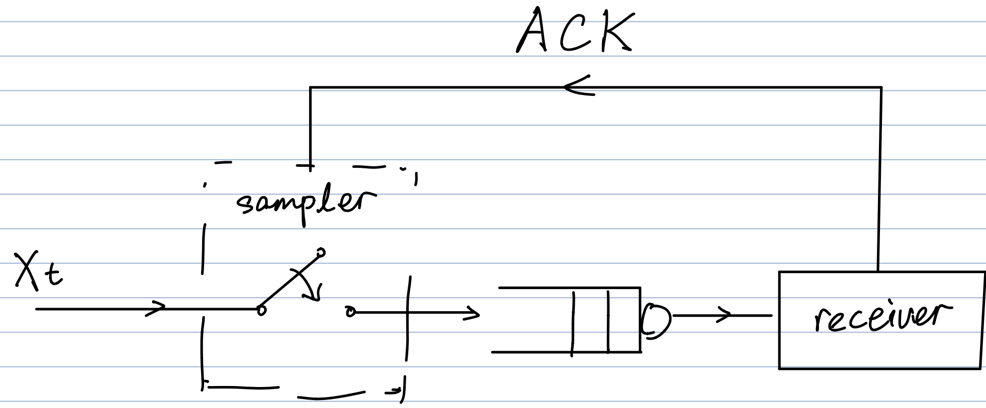
AoI and Sampling

Reading: Wait or Update TIT 2017.

JSAC AoI survey

Sun, Cyr 2019.

Ornee, Sun 2020



sample i: generation time S_i delivery time D_i

$$\Delta(t) = t - \max\{S_i : D_i \le t\}.$$

ACK: zero feedback delay.

server idle/busy state is known at the sampler.

- Consider a FCFS queueing system, with a general service time distribution.
- service time. Y_i. i.i.d.

$$0 < E[Y_i] < \infty.$$

$p(t)$ is non-decreasing.

$$\bar{P}_{opt} = \inf_{\pi \in \Pi} \limsup_{T \to \infty} \frac{1}{T} E\left[\int_0^T P(\Delta(t))\,dt\right].$$

$\pi: (S_1, S_2, \cdots)$ is a sampling policies

$\downarrow$

Sampling time.

Π: the set of causal policies.

Theorem:

The policy $(S_1(\beta), S_2(\beta), \cdots)$ defined by

$$S_{i+1}(\beta) = \inf\{t \geq D_i(\beta) : E[p(\Delta(t+Y_{i+1}))] \geq \beta\}.$$

where $D_i(\beta) = S_i(\beta) + Y_i$, $\Delta(t) = t - S_i(\beta)$,

and β is the root of

$$E\left[\int_{D_i(\beta)}^{D_{i+1}(\beta)} P(\Delta(t))\,dt\right] = \beta E[D_{i+1}(\beta) - D_i(\beta)]. \qquad (*)$$

is an optimal solution to (41).

Further $\beta = \bar{P}_{opt}$.

Sample $i+1$ is generated at the earliest time t satisfying:

(i) $t \geq D_i(\beta)$, sample i has been delivered.

(ii) $E[p(\Delta(t+Y_{i+1}))]$ has grown to $\geq$ threshold $\beta = \bar{p}_{opt}$.

Note: (*) has a unique solution.

Reading: Theorem 1
Sun, Cyr 2019.

Theorem 4.

Ornee, Sun 2020

Proof.

Step 1.

Lemma 1. It is sub-optimal to take a new sample before the previous sample is delivered.

Proof. If a sample is taken when the server is busy, one can design a better sampler, by postponing the sampling time to a later time when the server becomes idle. $\boxed{\checkmark}$

By this lemma, we only need to consider the policies

$$\Pi_1 = \{\pi \in \Pi : S_{i+1} \geq D_i = S_i + Y_i \text{ for all } i\}.$$

Let $Z_i = S_{i+1} - D_i \geq 0$. be the waiting time between D_i and S_{i+1}.

Designing the sampling times $(S_1, S_2, \ldots)$ is equivalent to designing the waiting times $(Z_1, Z_2, \ldots)$.

Step 2.

Suppose that $\forall \pi \in \Pi_1$,

(a) $$\limsup_{T\to\infty} \frac{1}{T} E\left[\int_0^T p(\Delta(t))\,dt\right] = \lim_{i\to\infty} \frac{E\left[\int_0^{D_i} p(\Delta(t))\,dt\right]}{E[D_i]}.$$

(b) $\lim_{i\to\infty} \frac{1}{i} E[D_i]$ $\lim_{i\to\infty} \frac{1}{i} E\left[\int_0^{D_i} p(\Delta(t))\,dt\right]$ exist.

The assumptions will be discussed later.

Then, we want to minimize.

$$\limsup_{T\to\infty} \frac{1}{T} E\left[\int_0^T p(\Delta(t))\,dt\right]$$

$$= \lim_{i\to\infty} \frac{E\left[\int_0^{D_i} p(\Delta(t))\,dt\right]}{E[D_i]}.$$

$$= \lim_{i\to\infty} \frac{\sum_{j=1}^{i} E\left[\int_{D_i}^{D_{i+1}} p(\Delta(t))\,dt\right]}{\sum_{j=1}^{i} E[Y_j + Z_j]}$$

$$\begin{cases} D_i = S_i + Y_i \\ S_{i+1} = D_i + Z_i \\ D_{i+1} = S_{i+1} + Y_{i+1}. \end{cases}$$

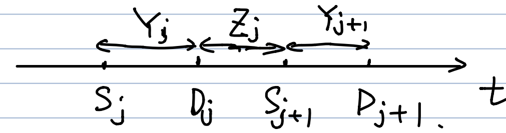
$$\Delta(t) = t - \max\{S_i : D_i \le t\}.$$

because $D_i \le D_{i+1}$ $\forall i$, then

$$\Delta(t) = t - S_i \quad \text{if } D_i \le t < D_{i+1}.$$

$$\int_{D_j}^{D_{j+1}} P(\Delta(t))\, dt$$

$$= \int_{D_j}^{D_{j+1}} P(t - S_j)\, dt.$$



$\tau = t - S_j$.

$$= \int_{Y_j}^{Y_j + Z_j + Y_{j+1}} P(\tau)\, d\tau$$

if $t = D_j$. then

$\tau = t - S_j = Y_j$.

if $t = D_{j+1}$, then

$\tau = t - S_j$

$= D_{j+1} - S_j$

$= Y_j + Z_j + Y_{j+1}$.

Def: $q(y, z, y')$

$$= \int_{y}^{y+z+y'} P(\tau)\, d\tau.$$

The problem is reformulated as

$$\bar{P}_{opt} = \inf_{\pi \in \Pi_1} \lim_{i \to \infty} \frac{\sum_{j=1}^{i} E[q(Y_j, Z_j, Y_{j+1})]}{\sum_{j=1}^{i} E(Y_j + Z_j)}. \qquad (1).$$

Step 3.

Consider the following problem:

$$h(c) = \inf_{\pi \in \Pi_1} \lim_{i \to \infty} \frac{1}{i} \sum_{j=1}^{i} E[q(Y_j, Z_j, Y_{j+1}) - c(Y_j + Z_j)]. \qquad (2).$$

Lemma. (a) $\bar{P}_{opt} \gtreqless c$ if & only if $h(c) \gtreqless 0$

(b) If $h(c) = 0$. then the solutions to (1) and (2) are identical.

Proof: "If $\bar{P}_{opt} \le c$, then $h(c) \le 0$".

If $\bar{P}_{opt} \le c$, then for any $\varepsilon > 0$.

there exists a policy $\pi = (Z_1, Z_2, \cdots)$ satisfying

$$\lim_{i\to\infty} \frac{\sum_{j=1}^{i} E[q(Y_j, Z_j, Y_{j+1})]}{\sum_{j=1}^{i} E(Y_j + Z_j)} \le c + \varepsilon.$$

$$\lim_{i\to\infty} \frac{\sum_{j=1}^{i} E[q(Y_j, Z_j, Y_{j+1})] - c\sum_{j=1}^{i} E[Y_j + Z_j]}{\sum_{j=1}^{i} E(Y_j + Z_j)} \le \varepsilon.$$

$E[Y_j] > 0$.

$$\frac{1}{i}\sum_{j=1}^{i} E(Y_j + Z_j) \ge m > 0.$$

$$\lim_{i\to\infty} \frac{1}{i}\sum_{j=1}^{i} E[Y_j + Z_j] \text{ exist}.$$

$$\lim_{i\to\infty} \frac{1}{i}\sum_{j=1}^{i} E[q(Y_j, Z_j, Y_{j+1})] - c\frac{1}{i}\sum_{j=1}^{i} E[Y_j + Z_j]$$
$$\le \varepsilon \frac{1}{i}\sum_{j=1}^{i} E[Y_j + Z_j]$$

if $\lim_{i\to\infty} a_i = a$. $\lim_{i\to\infty} b_i = b$. then

$\lim_{i\to\infty} \frac{a_i}{b_i} = \frac{a}{b}$. $\lim_{i\to\infty} a_i b_i = ab$.

The choice of $\varepsilon > 0$, is arbitrary.

$$\inf \left\{ \lim_{i\to\infty} \frac{1}{i} \sum_{j=1}^{i} E[q_r(Y_j, Z_j, Y_{j+1})] - c \frac{1}{i} \sum_{j=1}^{i} E[Y_j + Z_j] \right\}$$

≤ 0.

$h(c) \leq 0$.

" If $h(c) \leq 0$. then $\bar{P}_{opt} \leq c$ ".

Next we need

$\bar{P}_{opt} < c$ if & only if $h(c) < 0$.

$\bar{P}_{opt} \geq c$ if & only if $h(c) \geq 0$.

By this,

" $\bar{P}_{opt} \underset{\geq}{\overset{\leq}{=}} c$ if & only if $h(c) \underset{\geq}{\overset{\leq}{=}} 0$. "